NHH
INSTITUTT FOR FORETAKSØKONOMI
DEPARTMENT OF FINANCE AND MANAGEMENT SCIENCE

FOR 8 2009
ISSN: 1500-4066
SEPTEMBER 2009

Discussion paper

Approximating Closed Form Solutions to a Class of Feedback Policies

BY
LEIF K. SANDAL

Norges Handelshøyskole
NORWEGIAN SCHOOL OF ECONOMICS AND BUSINESS ADMINISTRATION

Approximating Closed Form Solutions to a Class of Feedback Policies

Leif K. Sandal

June 18, 2009

Abstract

Dynamic optimization problems cover a large class of problems in theoretical and applied economics. A simple iterative algorithm with fast convergence is proposed. It is demonstrated that the algorithm in a few steps produce excellent analytic (closed form) approximations including error bounds to a class of nonlinear problems.The algorithmic scheme is also well suited to produce numerical solutions. The notions of **dynamic** and **potential** rents are operationalized. The algorithm is utilizing a relation balancing these concepts. The result is particularly strong in the case of zero discounting where the exact CU-optimal policy is determined in a single step.

Applying a particular seed in the general convergent scheme reproduces in a simple way results (formulas) published in the last decade in bioeconomics.

JEL classification: A12, C61, C63, E10, Q00

Keywords: Closed form approximations; Contraction algorithm; Renewable resource economics; Capital dynamic modeling; Zero discounting and optimality

1 Introduction

A large class of theoretical and applied research problems can be modeled by a dynamic optimization process. The classical formulations, like Hamilton's principle in mechanics, being the variational approach and the optimal control theory in economics like the Pontryagin's maximum principle or the general dynamic programming approach.

In this account the exploitation of a renewable capital stock is rewritten into a form suitable for marginal contribution interpretation by defining two new mappings (M and N) which we label as dynamic rent and potential rent. The feasible policy that balances these two rents represents the optimal policy in feedback form. Moreover, the balancing equation gives rise to a contraction mapping. A simple iterative algorithm is shown to converge rapidly to a solution. A remarkably simple error control is also provided. The algorithm can be used as a tool to obtain closed form (analytical) feedback approximations. It is also efficient in obtaining numerical solutions.

We start with the mathematical formulation of the model in section two. The main result is also presented in this section. The account given is kept brief. Further explanations and details needed to fully appreciate the ideas involved are given in appendices A and B for the sake of readability. Basic definition, assumptions and proofs are laid out in an natural way in these appendices to avoid cluttering the main exposition with technicalities. In section three we revisit two real world examples from the literature and demonstrate that our new solution procedure is working very well. The theorem derived is constructive and can be used directly to produce numerical as well as analytical approximations. Moreover, it has a nice property in the limit of vanishing discounting. In this limit it produces the exact (CU-) optimal[1] solution in a single "iteration".

The modeling approach and results presented in this paper are, however, of a more general character than the special field of application we focus on here i.e. ideas from natural renewable resource management[2].

The main part of paper is divided into 4 sections followed by two appendixes. Following the introduction, section 2 establishes the model. It outlines the feedback policy setting and introduces the main concepts of dynamic and potential rents and their balancing equation that constitutes the key contraction relation for iterations with given error bounds. In section 3 we apply the theory on real world bioeconomic examples from the fishery management literature and demonstrate that our new approach works well numerically as well as analyt-

[1] See Seierstad and Sydsæter (1999) for a definition of CU-optimality.

[2] See e.g. Arnason, Sandal, Steinshamn, and Vestergaard (2004); Sandal and Steinshamn (2001a, 2000); Grafton, Sandal, and Steinshamn (2000); Sandal and Steinshamn (1998) and Clark (1990).

ically. It generalizes earlier results found and provide well-defined error bounds where they were not previously available. Section 4 is a brief summary and discussion. Key features are a simple scheme, fast contraction[3] and error control.

2 The Model

In managing renewable resources the following class of problems is of interest. Obtain

$$\max_{\tilde{u}\in U} \int_0^\infty e^{-\delta t}\Pi(x(t),\tilde{u}(t))dt\,, \tag{1}$$

where $x \in X$, $\tilde{u} \in U$ and X, U are intervals. Here Π is a twice differentiable and strictly concave function of $\tilde{u}$. The discount rate δ is nonnegative. To avoid unnecessary technicalities it is sufficient in our context to assume that $x = x(t)$ and $\tilde{u} = \tilde{u}(t)$ are *piecewise smooth*[4] and $x = x(t)$ is continuous. The maximization is to be performed subject to the constraining state equation[5]

$$\dot{x} \stackrel{def}{=} \frac{dx}{dt} = f(x) - \tilde{u}\,. \tag{2}$$

Our main objective is not to derive existence or uniqueness results but to provide a simple scheme for getting closed form approximations to a class of nonlinear real-world problems. We may assume that some criteria for existence of an optimal solution are satisfied, e.g. that of the Mangasarin sufficiency and uniqueness theorem for the Catching-Up optimality[6]. In this work we take for granted the existence of at least one optimal policy and we focus solely on conditions ensuring our approach to pick an optimal control path in state and policy space. In the problems we have in mind x may be interpreted as a measure of the stock level of a renewable resource, $\tilde{u}$ the harvest rate and $f(x)$ the natural surplus growth rate of the resource.

In the next section we present an alternative formulation by introducing the notions of dynamic and potential rent. These quantities play a similar role in our problem as potential and kinetic energy do in classical mechanics.

[3]Only few iterations are needed for good approximations.

[4]The term is short for functions with a continuous derivative except at a finite number of points.

[5]The approach in this paper can be generalized to a larger class of state equations. We have chosen a simple form to avoid cluttering the presentation and interpretations with too many technicalities.

[6]See e.g. theorem 13 on page 234 in Seierstad and Sydsæter (1999).

2.1 The Feedback Approach

Here we consider the state and control space and treat time t as a redundant parameter. In this space we examine piecewise continuously differential functions $u = u(x)$, usually called policies in feedback form[7] or control paths. We allow for discontinuities in the derivative as well as in the function itself at a finite number of points. The model can be formulated as an integral equation

$$M(x,u) = \tilde{N}(x)[u,\alpha]\,, \tag{3}$$

where

$$M(x,u) \stackrel{def}{=} \Pi(x,u) + \Pi_u(x,u)\,\{f(x) - u\} - S(x)\,, \tag{4}$$

and

$$S(x) \stackrel{def}{=} \Pi(x, f(x))\,. \tag{5}$$

Furthermore

$$\tilde{N}(x)[u,\alpha] \stackrel{def}{=} K(\alpha) - S(x) + \delta \cdot \int_\alpha^x \Pi_u(s,u(s))ds\,, \tag{6}$$

where α and $K(\alpha)$ are parameters to be determined. We have by definition that $M(x, f(x)) \equiv 0$. A derivation of Eq. (3) is offered in appendix A. The equation in simplified form reads

$$\Pi(x,u) + \{f(x) - u\} \cdot \Pi_u(x,u) = K(\alpha) + \delta \cdot \int_\alpha^x \Pi_u(s,u(s))ds\,. \tag{7}$$

The formulation given in Eq. (3) is preferred due to the nice properties held by the operators M and $\tilde{N}$. Before establishing these properties we give some interpretations of the different terms involved. Notice that M is an explicit given function while $\tilde{N}$ is a functional depending on the state x, an interval in state space and a policy function u over the interval. More details concerning the properties of M and $\tilde{N}$ are stated and proved through a series of propositions and lemmas in appendix B. In the present setting Eq. (2) is actually redundant and only serves the purpose of connecting the problem to the time domain. Interpretations of the terms in Eq. (3) within a typical economic framework are the time rates

$$\begin{aligned} \Pi &\sim \text{rent},\\ (f-u)\cdot\Pi_u &\sim \text{saving/investments},\\ S(x) &\sim \text{sustainable rent},\\ K(\alpha) &\sim \text{rent in a reference state}, \end{aligned}$$

[7] By definition $u = u(x(t)) = \tilde{u}(t)$ and hence $u = u(x)$ is the feedback form of the policy.

$$\delta \cdot \int_{\alpha}^{x} \Pi_u(s, u(s))ds \quad \sim \quad \text{running cost or gain in moving between } \alpha \text{ and x.}$$

We observe that M is made up of the first three terms in the above list where the two first terms represent the net total gain. The third term represents sustainable rent, that is the rent obtained by freezing the situation, hence M can be interpreted as the net gain by changing the state and we refer to it as *dynamic rent*. It follows directly from the definition of M and the concavity of Π that the dynamic rent is non-negative. On the other hand $\tilde{N}$ is made up by the last three terms on the list and is thereby a measure of the net accumulated cost of changing the state by following a particular path and we refer to it as the *potential rent*.

We have that $M = \tilde{N}$ is the point of balance in marginal contributions in time. Thus the optimal policy corresponds to the case where we have a decreasing marginal contribution by following an optimal policy in time, i.e. $M \to 0$ with time as we are approaching a stationary state. It is seen from the definition of the dynamic rent that it resembles the notion of kinetic energy in physics by noticing that

$$M(x, u) = -\Pi_{uu}(x, u)(f(x) - u)^2 + o((f(x) - u)^2) = \frac{1}{2}m\dot{x}^2 + o(\dot{x}^2),$$

and $m = -2\Pi_{uu}(x, u) > 0$ and $\dot{x}$ represent the mass and velocity of a rigid body. Thus our problem is similar to a physical situation where an object is moving towards an equilibrium position in a manner such that the kinetic energy $E_k = \frac{1}{2}mv^2$ associated with the dynamics is decreasing over time. We are looking for a position to end motion by asking: Which states (α-values) are possible "parking" spots? In analogy with the physical picture it will be the states with extreme potential energy which in our setting implies a state which maximize the potential rent subject to being at rest or fixed. Restricting ourselves to an inner optimum we obtain

$$\alpha = \arg\max_{\alpha \in X} \tilde{N} \quad \Rightarrow \begin{cases} K(\alpha) = S(\alpha) = \Pi(\alpha, f(\alpha))\,, \\ K'(\alpha) = \delta \cdot \Pi_u(\alpha, u(\alpha)) = \delta * \mu(\alpha)\,, \\ u(\alpha) = f(\alpha)\,, \end{cases}$$

implying

$$S'(\alpha) = \delta \cdot \mu(\alpha) \quad \text{where} \quad \mu(x) \overset{def}{=} \Pi_u(x, f(x))\,. \tag{8}$$

It has been shown by Sandal and Steinshamn (1997b) that relation (8) is the general equilibrium condition. The term $S'(\alpha)$ is the marginal change in the sustainable rent due to a marginal change in stock/state level and $\delta\mu(\alpha)$ represents the alternative rate of return this change can earn. Eq. (8) is a compact form of the equilibrium relation in natural renewable resource economics and it is sometimes referred to as the Golden Rule. It simply states that the marginal value added by not utilizing a unit of the resource is equal to the marginal gain of utilizing it. If such a resource level exists, the management regime will be indifferent whether to utilize the resource unit or not and hence the resource level becomes a settling down state.

In the present context we assume that such a state exists and is determined by solving Eq. (8). It is chosen to be our point of reference. We label it $\alpha = x^*_\delta$. If there is more than one solution we choose one that maximize S. Notice that it is sufficient that the scalar function $\tilde{S}(x) = S(x) - \delta \int \mu(x)dx$ is strictly concave for a unique equilibrium to exist. See Sandal and Steinshamn (1997b) for further details.

2.2 Main Result

In order to present the main result in a compact way we list some key definitions and assumptions.

The playground in state and control space is A^* given by

$$A^* \stackrel{def}{=} \{(x,u) \mid (x - x^*_\delta)(f(x) - u) \leq 0, \quad x \in X, \quad u \in U\}\,, \tag{9}$$

Paths restricted to the set A^* represents policies that do not move the state away from its optimal stationary level. The following assumption is expected to hold throughout the rest of the paper:

A1 *The function $\Pi(x,u)$ is twice differentiable in its arguments with $\Pi_u \geq 0$ and*

$$0 < m_1(x) \leq -\Pi_{uu}(x,u) \leq m_2(x) < \infty \quad \text{on} \quad (x,u) \in A^*\,,$$

for a pair of integrable functions m_1 and m_2.

We also need the following relations

$$\begin{aligned} M(x,u) &= \Pi(x,u) + \Pi_u(x,u)\{f(x) - u\} - S(x) \\ N(x,u) &\stackrel{def}{=} S^*_\delta - S(x) + \delta \cdot \int_{x^*_\delta}^{x} \Pi_u(s,u(s))ds = \tilde{N}(x)[u, x^*_\delta]\,, \end{aligned} \tag{10}$$

$$S^*_\delta \stackrel{def}{=} S(x^*_\delta) = \Pi(x^*_\delta, f(x^*_\delta))\,, \tag{11}$$

Definition 1 *Admissible Controls*
Admissible controls (feedback policies) are $u = u(x) \in PS(X)$ where $PS(X)$ is the set of

piecewise smooth and bounded functions such that $(x, u(x)) \in A^*$.

A sequence is called an admissible if all the terms in the sequence are admissible controls.

We can now state our main result in the following theorem:

Theorem 1 *Convergence and Error Bounds*
Any infinite admissible sequence $\{v_n\}$ *produced by* $M(x, v_{n+1}) = N(x, v_n)$ *for an arbitrary seed* v_0*, converge to the unique admissible solution* u *satisfying* $M(x, u) = N(x, u)$*. Moreover, if* $\{v_n\}$ *is a super-sub sequence* (e.g. $v_0 = f(x)$) then $u = \frac{1}{2}(v_{n-1} + v_n) + \epsilon$ where $|\epsilon| \leq \frac{1}{2}|v_n - v_{n-1}|$.

The proof of this theorem is rather tedious. Details are given in Appendix B. The core in the proof is the construction of enveloping sequences that are alternating around the unique solution (sub-super solutions). This construction is reflected in the error bound given. The theorem is constructive in the sense that one can choose freely a seed function or a collection of numerical points representing the function and start iterating. Any kind of proper numerical integrator can be used in evaluating the right hand side in the iteration scheme if one wishes to find numerical approximations/solutions. We will here emphasize that the theorem gives an easy way to find analytic bounds on the solution and thereby analytical (closed form) approximations to the solution.

Solutions are particularly easy to find when $\delta \rightarrow 0$.[8] In this limit equation (7) becomes the ordinary equation $M(x, u) + S(x) = S_0^*$ for the feedback policy u(x). Formally this limit can be interpreted as a generalized optimality. That is, even though different policies may create infinite large utilities there are no value of a discount rate that can alter the practical result that "two dollars a day is better than one dollar a day for infinitely long time". Catching-Up optimality (CU-optimality) is a generalization along theses ideas. This and other natural extensions of the notion of optimality can be found in e.g. Seierstad and Sydsæter (1999). The formal solution of this equation is in fact a separatrix in phase space going through the reference point x_0^*. The equation itself is a convenient way of stating the fact that the Hamiltonian is conserved in this special case.

[8]The zero discounting case is problematic in classical numerical infinite horizon approaches. The CU-optimal feedback policy limit is also hard in modern viscosity solution approach. See e.g. Fleming and Soner (1993) on page 371.

3 Examples from Fisheries Management

In this section we exemplify the strength and applicability of our main result by reproducing some published results. Both examples are illustrations of the potential power of theorem 1. We revisit two important bioeconomic cases in fishery management. Both cases concern the problem of determining the total allowable catch (TAC) quotas. Our main result is by no means limited to this class of problem. Among others, the papers, Sandal and Steinshamn (1998, 2000, 2001b) and McDonald, Sandal, and Steinshamn (2002) deal with models that can be suitably handled with the procedure described in this paper.

3.1 A Northern Cod Fishery Model

This is a non-trivial illustration of the potential our procedure has to offer by producing good analytical results without specifying a model in full details. In Grafton, Sandal, and Steinshamn (2000) the collapse of the Canadian Northern Cod Fishery was investigated by using a model defined by the economic relations

$$\Pi(x,u) = P(u)\,u - C(x,u)\,, \quad P(u) = \frac{a\,p_1 + u\,p_0}{a+u}\,, \quad C(x,u) = q\frac{u}{x}\,. \tag{12}$$

The prices p_0 and p_1 are the minimum and the maximum prices, a the flexibility parameter in the inverse demand function and q is the derived cost parameter measuring the cost per unit output per unit biomass. The biomass is assumed to be updated according to

$$\dot{x} = f(x) - u = r\,x\,(1 - \frac{x}{K})^{\alpha} - u \tag{13}$$

where r is the intrinsic growth rate for the biomass and K is the stock's carrying capacity in terms of biomass. The parameter α is measuring how much the maximum sustainable yield (MSY) is skewed to left or right of the MSY in the logistic model (at $x = 0.5K$ for $\alpha = 1$). The problem is naturally restricted to the region $x = [0, K]$. Applying the appropriate definitions we get

$$\begin{aligned}
M(x,u) &= \frac{a^2(p_1-p_0)}{a+f(x)} \cdot \left[\frac{(f(x)-u)}{(a+u)}\right]^2 , && (14)\\
S(x) &= \Pi(x, f(x)) = (a\,p_1 + f(x)\,p_0)\,\frac{f(x)}{a+f(x)} - q\frac{f(x)}{x}\,, && (15)\\
\Pi_u(x,u) &= p_0 - \frac{q}{x} + \frac{(p_1-p_0)a^2}{(a+u)^2}\,, && (16)
\end{aligned}$$

$$
\begin{aligned}
\mu(x) &= \Pi_u(x, f(x))\,, \\
N(x,u) &= S(x^*) - S(x) + \delta\, p_0(x - x^*) - \delta\, q\, \ln\left(\frac{x}{x^*}\right) \\
&\quad + \delta\,(p_1 - p_0)\, a^2 \int_{x^*}^{x} \frac{ds}{(a + u(s))^2} \\
&= \hat{S}(x) + \delta\,(p_1 - p_0)\, a^2 \int_{x^*}^{x} \frac{ds}{(a + u(s))^2}\,.
\end{aligned} \tag{17}
$$

The reference state x^*_δ is the solution of $S'(x) = \delta\,\mu(x)$. It is worth pointing out that we have an exact formula for the feedback policy in the limit of vanishing discounting. In this case we must interpret the optimality in a generalized sense, e.g., the CU-optimality[9]. The iteration scheme $M(x, u_{n+1}) = N(x, u_n)$, can be started with the static optimal policy making $\Pi_u(x, u_0) = 0$, which is the same as neglecting the discount rate in the first iteration. Thus it gives the exact discount free solution. We get explicit closed form expression for $\{u_0, u_1\}$ from

$$
\Pi_u(x, u_0) = p_0 - \frac{q}{x} + (p_1 - p_0)\frac{a^2}{(a + u_0)^2} = 0
$$

and

$$
\hat{S}(x) = S(x^*) - S(x) + \delta \cdot p_0 \cdot (x - x^*) - \delta \cdot q \cdot \ln(x/x^*) = \frac{a^2(p_1 - p_0)}{a + f(x)} \cdot \left[\frac{f(x) - u_1}{a + u_1}\right]^2
$$

The second iteration is for all practical purposes the solution. This is the same type of solution we get if we use classical perturbation theory with the discount rate as the perturbation parameter, see Sandal and Steinshamn (1997a,c). In our setting it is just a result of a particular choice of seed in our general iteration scheme[10]. Notice that the system becomes singular when $p_1 = p_0$, signaling that only the equilibrium point fits the equation. This is in fact what we will expect since we then know that the optimal behavior is a bang-bang policy with the switch at the equilibrium point. The *CU*-optimal policy in the limit $\delta \to 0$ is given by $u = u_1$,

$$
u_1(x) = \frac{a\Lambda(x) + f(x)}{1 - \Lambda(x)}\,, \quad \Lambda(x) \overset{def}{=} \operatorname{sgn}(x - x^*_\delta)\sqrt{\frac{(a + f(x))\hat{S}(x)}{a^2(p_1 - p_0)}}\,, \tag{18}
$$

where the signum function stems from the definition of the region A^*.

[9]See Seierstad and Sydsæter (1999) for different extensions of the notion of optimality in the infinite horizon case.

[10]An even better approximation is produced by using $u_0 = k\,x$ as seed, a line through two known points on the true solution if $k = \frac{f(x^*_\delta)}{x^*_\delta}$.

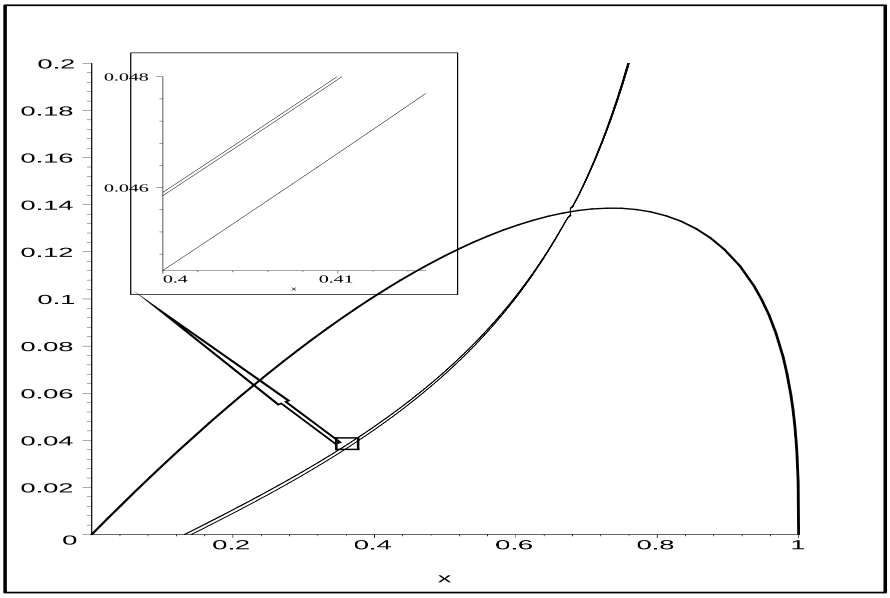

Figure 1: *This figure shows the function $f(x) = u_0$ (starts at $(0,0)$ and ends at $(1,0)$, and the approximate optimal harvests u_1, u_2, u_3. The common point for all curves is $(x^*_\delta, u(x^*_\delta))$, the equilibrium point. Absolute error bounds after 1,2,3 iterations:* 0.058, 0.00167, 0.0000577. *Discount rate is 5%. Considering the curves for $x < x^*_\delta$, the lower curve is u_1, the upper curve is u_2 and u_3 is the middle curve. This can be seen more clearly in the window where a short portion of these curves are enlarged. The unique solution u^* is located in between u_2 and u_3. This demonstrates that $u = \frac{1}{2}(u_2 + u_3)$ is a very good approximation. Stock and harvest are scaled by the carrying capacity (K).*

The above seed in the iterative process is only worth working with if we need a formula for the solution. In the numerical analysis we can start with any function in the relevant region. However, $u_0 = f(x)$ is an excellent choice since it is a super-sub solution.[11] By applying the parameter values from Sandal and Steinshamn (1997a) $r = 0.3036$, $K = 3.2 \times 10^6$, $\alpha = 0.3587$, $p_{min} = 200$, $p_{max} = 1250$, $a = 0.139 \times 10^6$, $q = 2.006 \times 10^8$. We have plotted the iterative solutions $\{u_0, u_1, u_2, u_3\}$ for the case $\delta = 5\%$.

3.2 Bluefin Tuna

In the article McDonald, Sandal, and Steinshamn (2002), a bioeconomic model for Bluefin Tuna in the Southern hemisphere is investigated. We illustrate our technique by applying it

[11] A super-sub solution is any function that is a super solution to the left of $x = x^*_\delta$, and a sub solution to the right, i.e. it is not below the true solution to the left and not above it to the right of $x = x^*_\delta$.

on this model. We summarize the basics in the following relations

$$\Pi(x,u) = \gamma(x)\, u - \Gamma(x)\, u^2 - \alpha(x)\,, \quad \dot{x} = f(x) - u\,. \tag{19}$$

The *dynamic* and *potential* rates are

$$\begin{aligned}
M(x,u) &= \Gamma(x)(u - f(x))^2\,,\\
\Pi_u(x,u) &= \gamma(x) - 2\,\Gamma(x)\, u\,,\\
\mu(x) &= \gamma(x) - 2\,\Gamma(x)\, f(x)\,,\\
S(x) &= \gamma(x)\, f(x) - \Gamma(x)\,(f(x))^2 - \alpha(x)\,,\\
N(x,u) &= S(x_\delta^*) - S(x) + \delta \int_{x_\delta^*}^{x} \gamma(s)ds - 2\delta \int_{x_\delta^*}^{x} \Gamma(s)u(s)ds\\
&= \hat{S}(x) - 2\delta \int_{x_\delta^*}^{x} \Gamma(s)u(s)ds\,.
\end{aligned}$$

Hence the iterations can be explicitly calculated, giving

$$u_{n+1}(x) = f(x) + \operatorname{sgn}\,(x - x_\delta^*) \sqrt{\frac{\hat{S}(x)}{\Gamma(x)} - \frac{2\delta}{\Gamma(x)} \int_{x_\delta^*}^{x} \Gamma(s)\, u_n(s)ds}\,. \tag{20}$$

Using the myopic policy (the infinite discounting solution) as seed, i.e. $u_0 = \gamma(x)/(2\,\Gamma(x))$, we get the exact zero-discounting solution in the first iteration which also was the case in the former example. This behavior is a general property of our procedure.

In the appendix we prove the general statement that any super (sub) seed generates a sequence alternating between super and sub solutions. In the above example we started with a tight super solution (myopic or static policy) and next iterate is a tight sub solution (exact zero discount solution). It should be rather obvious from a practical interpretation of the problem that the myopic harvest is larger than the first best and the zero discounting harvest is too conservative.

We state the basic input functions for the Bluefin Tuna problem given in McDonald, Sandal, and Steinshamn (2002). The fish stock is modeled by a surplus growth function with critical depensation[12]

$$\Pi = a \cdot u - b \cdot u^2 - \frac{d}{x}\,, \quad f(x) = r \cdot (x - k) \cdot (1 - \frac{x}{K})\,, \tag{21}$$

where the following parameter values are used: $r = 0.2246$, $K = 0.565 \times 10^6$, $a = 88.25$,

[12] Critical depensation implies that the stock goes extinct by itself if the stock falls below a given level.

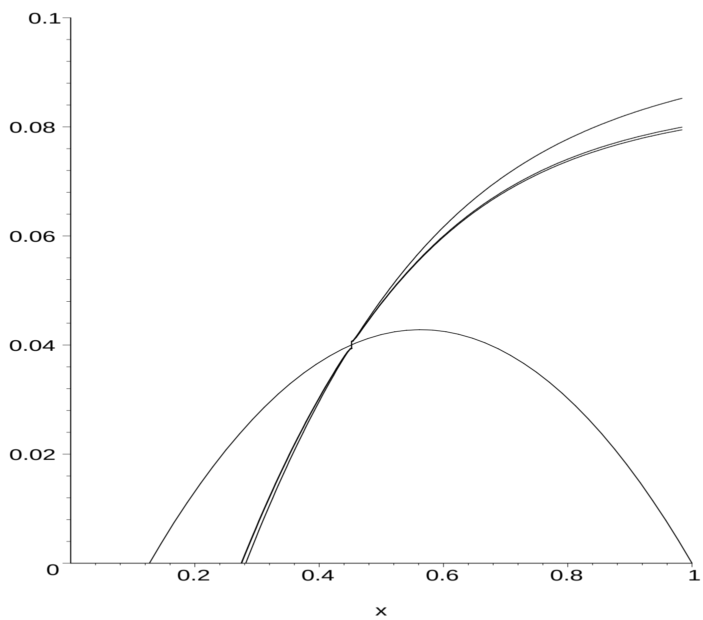

Figure 2: *This figure shows the function* $u_0 = f(x)$ *(ends at* $(1, 0)$*), and the approximate optimal harvest* u_1, u_2, u_3. *The common point for all curves is* $(x^*_\delta, u(x^*_\delta))$, *the equilibrium point. Absolute error bounds after 1,2,3 iterations are:* 0.041, 0.00289, 0.000246. *Discount rate was set to 5%. Considering the curves for* $x > x^*_\delta$, *the upper curve is* u_1 *and* u_2 *is the lowest curve. Furthermore* u_3 *is the middle curve. The unique solution* u^* *is located in between* u_2 *nd* u_3. *This demonstrates that* $u = \frac{1}{2}(u_2 + u_3)$ *is a very good approximation. Stock and harvest are scaled by the carrying capacity (K).*

$b = 9.0 \times 10^{-4}$, $d = 1.63 \times 10^{11}$, $k = 0.716 \times 10^6$. Results are plotted in figure 2. The approximate optimal solutions displayed in figures 1 and 2 are in excellent agreement with the cited references.

4 Summary

We have presented a novel procedure to solve a class of non-linear infinite horizon control problems. By introducing the new general notions of *dynamic* rent and *potential* rent we obtain an equation for the optimal feedback policy by balancing the two types of rents. This balancing equation gives rise to a contraction mapping with extra nice properties that can be utilized to make a robust iteration scheme for determining numerically as well as analytically the feedback policy with explicit error bounds. In the case of vanishing discounting (the autonomous case) the balancing of the *dynamic* and *potential* rent yields and ordinary (algebraic) equation for the Catching-Up optimal feedback policy.

The case with a single state and control is important. Lumped variables are often the best approach to many applications in bioeconomics because the policy tools in many circumstances are rough and on an accumulated level. This simplification may be a result of practical limitations on how the policy can be implemented. A lot of fisheries around the world are managed by setting the total annual catch (TAC). It is no point in having models cluttered with more details than what are present in the real decision process on the real decision level. The art of modeling is closely related to the ability to make models as simple as possible and still capture the main features set out to be explored. It is a typical pitfall to try to model as much as possible in a single approach. The result is loss of clarity and causality. It becomes impossible to anchor the model in a proper way to the real world. There is simply too much uncertainty floating around and too many parameters that can not be resolved from data. In any case it is a good approach to check and evaluate more complex models by how they perform compared to proper (preferable analytical) solutions of some non-trivial degenerated cases of the model. Our approach picks up nonlinear effects in an easy and transparent way. Thus we believe that our results can be of some value.

In the examples presented from the current literature, a few iterations are sufficient to obtain an approximate solution of very high quality and accuracy. This is demonstrated in a striking way in figures 1 and 2.

There are many interesting features in our formulation. The economic interpretations of M as dynamic rent and N as potential rent and their properties are the most prominent. Another basic feature is the source term in the iterations scheme, N (the integral term), which is the only place where the discount rate, δ, enters explicitly. The fact that the discount rate is typically a small quantity suggests that the exact solution for $\delta = 0$ contains the proper behavior that stems from the nonlinearity in the problem. This policy is straight forward to obtain in our scheme ("a single iteration"). A small discount rate is mainly shifting the policy slightly upwards. This statement does of course only hold for the policy. The value function itself may change dramatically, i.e. become infinitely large in the limit of zero discounting. In classical perturbational approach the discount rate may have a significant impact on the speed of convergence as has been established in the referenced papers. In the context of the present paper the classical perturbational (i.e an asymptotic series approach) is just a consequence of a particular choice of seed in a general procedure. Hence it is a way to check the quality of such asymptotic series.

The convergence rate can actually be investigated analytically and this analysis brings forward this feature. We do not include the details of this analysis in the present account because a numerical check of the accuracy is easily available due to simple error bounds. The fact that very few iterations produce a sufficiently accurate solution offers the possibility to analytically determine an approximate solution. This is a much more powerful tool for parameter adaption in any modeling procedure than any particular numerical solution can offer.

Finally, we believe that the procedure and results outlined in this paper can be carried over to the analogous finite horizon problems. By using the appropriate transversality conditions it should be possible two pick out a point on the optimal policy which then serves the role as our reference point and replaces $\alpha = x_\delta^*$ in the present exposition.

References

Arnason, R., L. K. Sandal, S. I. Steinshamn, and N. Vestergaard (2004): "Optimal Feedback Controls: Comparative Evaluation of the Cod fisheries in Denmark, Iceland and Norway," *American Journal of Agricultural Economics*, 86(2), 531–542.

Bardi, M., and I. C. Dolcetta (1997): *Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations*, System Control. Birkhauser Boston Inc, Boston.

Clark, C. W. (1990): *Mathematical Bioeconomics The Optimal Management of Renewable Resources*. New York, John Wiley & Sons, New York.

Fleming, W. H., and H. M. Soner (1993): *Controlled Markov Processes and Viscosity Solutions*. Springer-Verlag.

Grafton, R., L. K. Sandal, and S. I. Steinshamn (2000): "How to improve the management of renewable resources: The case of Canada's northern cod fishery," *American Journal of Agricultural Economics*, 82(3), 570–580.

McDonald, D., L. K. Sandal, and S. I. Steinshamn (2002): "Implications of a nested Stochastic/Deterministic Bio-Economic Model for a Pelagic Fishery," *Ecological Modelling*, 149(1-2), 193–201.

Sandal, L. K., and S. I. Steinshamn (1997a): "A feedback model for optimal management of renewable natural capital stocks," *Canadian Journal of Fisheries and Aquatic Sciences*, 54, 2475–2482.

——— (1997b): "Optimal Steady State and the Effect of Discounting," *Marine Resource Economics*, 12, 95–105.

——— (1997c): “A Stochastic Feedback Model for Optimal Management of Renewable Resources,” *Natural Resource Modeling*, 10, 31–52.

——— (1998): “Dynamic corrective taxes with flow and stock externalities: A feedback approach,” *Natural Resource Modeling*, 11, 217–239.

——— (2000): “The Cost of Underutilization of Labour in Fisheries,” *Annals of Operations Research*, 94, 1–13.

——— (2001a): “A Bio-economic model for Namibian Pilchard,” *The South African Journal of Economics*, 69(2), 1–24.

——— (2001b): “A simplified feedback approach to optimal resource management,” *Natural Resource Modeling*, 14(3), 419–432.

Seierstad, A., and K. Sydsæter (1999): *Optimal Control Theory with Economic Applications*, Advanced textbooks in Economics. North-Holland, Amsterdam, second edn.

A Balancing Dynamic and Potential Rents

A derivation of the relation, Eq. (3), is offered here. This equation is at the heart of our approach. It expresses the balance between the dynamic and potential rents that must be fulfilled by the optimal path.

Let H and λ be the current value representations of the Hamiltonian and the shadow price respectively. The Hamiltonian formulation results in the following definitions and system of equations

$$\begin{aligned} H(x,\tilde{u},\lambda) &\stackrel{def}{=} \Pi(x,\tilde{u}) + \lambda \cdot (f(x) - \tilde{u})\,, \\ \tilde{u} &= \arg\max_{\tilde{u}\in U} H\,, \qquad (H_{\tilde{u}} = 0 \text{ for internal optimum}) \\ \dot{x} &= f(x) - \tilde{u} = H_\lambda\,, \\ \dot{\lambda} &= \delta\,\lambda - H_x\,. \end{aligned}$$

A consequence of these equations is

$$\dot{H} = H_x\,\dot{x} + H_{\tilde{u}}\,\dot{\tilde{u}} + H_\lambda \dot{\lambda} + H_t = \delta\,\lambda\,\dot{x}.$$

For the cases studied the term $H_{\tilde{u}}\,\dot{\tilde{u}} = 0$, either because $H_{\tilde{u}} = 0$ (internal optimum) or otherwise because $\dot{\tilde{u}} = 0$. The equation above replaces the equation for the shadow price i.e. the equation for $\dot{\lambda}$. By integration in time and a substitution to the state and policy space we get

$$H = K + \delta \int_{t_0}^{t} \lambda\,\dot{x} dt = K + \delta \int_{\alpha}^{x} \lambda\,dx\,.$$

K is a constant of integration, which also represent the value of the optimal Hamiltonian at an arbitrary point in state and policy space on the optimal policy path. At an inner dynamical optimum we have $\lambda = \Pi_{\tilde{u}}$, and the result can be written as

$$H = K + \delta \int_{\alpha}^{x} \Pi_u(z, u(z))\,dz\,.$$

The feedback form of the policy, $u = \tilde{u}(t) = u(x(t)) = u(x)$, has been used. This is Eq. (7) which implies the main relation balancing dynamic and potential rents as expressed by Eq. (3).

Observe that we switched from integration over t to integration over x because the first order conditions for the current values are autonomous and t is a redundant parameter. We may consider $u = u(x)$ instead of $\tilde{u} = \tilde{u}(t)$, where $u(x)$ and $\tilde{u}(t)$ are entirely different functions, although connected through the fact that they have the same value at any given point in time. The equation for $\dot{x}$ is of no use in the state-control space. In a way we can formally think of x as being observed and $u(x)$ is our response to this observation. That is, we do not need the evolution equation for the "observable state" x in order to react in an optimal way. This is of course just a formal phrasing. We are dealing with a deterministic system and hence we can in principle alway calculate the path in state space from a given initial position. It is in this context we must interpret the statement that the $\dot{x}$-equation only serve to connect the (x, u)-space to the time domain. We are simply studying the optimal policy problem in phase space rather than investigating the trajectories parameterized by time. In this setting a proper phase space analysis is indeed a feedback description.

B Derivation of Main Result

This appendix contains definitions, lemmas and propositions leading to a proof of our main result as it is given in Theorem 1. We first state two important assumptions

Basic Assumptions

A1 *The function* $\Pi(x, u)$ *is twice differentiable in its arguments with* $\Pi_u \geq 0$ *and*

$$-\infty < -m_2(x) \leq \Pi_{uu}(x, u) \leq -m_1(x) < 0 \quad \text{on} \quad (x, u) \in A^*,$$

for some integrable functions m_1 *and* m_2.

A2 *The scalar function* $\psi(x)$ *is positive definite on* $x \in X$ *and is defined by*

$$\psi(x) \stackrel{def}{=} S^*_\delta - S(x) + \delta \int_{x^*_\delta}^{x} \mu(s)ds = S^*_\delta - \Pi(x, f(x)) + \delta \int_{x^*_\delta}^{x} \Pi_u(s, f(s))ds.$$

The function $\psi(x) \equiv N(x, f(x))$ has a critical point at $x = x^*_\delta$, which is stated in Eq. (8). We assume that this is representing a unique global minimum. Ψ plays the role of potential rent gain and the stationary state is a state with no potential gain with the interpretation that all potential rent is being continuously realized. Hence determination of the stationary state x^*_δ can be thought of as follows:

The preferred equilibrium state in a dynamic economic adjustment process is the state with all potentials utilize and is determined by finding the global minimum of the scalar function $\psi(x)$.

The following quantities play a key role in our proofs.

(22) $$Q(x,u) \stackrel{def}{=} M(x,u) - N(x,u) \quad \text{and} \quad (x,u) \in A^*$$

(23) $$A_L^* \stackrel{def}{=} \{(x,u)|\,(x,u) \in A^* \quad \& \quad x \le x_\delta^*\}\,,$$

(24) $$A_R^* \stackrel{def}{=} \{(x,u)|\,(x,u) \in A^* \quad \& \quad x \ge x_\delta^*\}\,.$$

Notice that $A_L^* \cup A_R^* = A^*$ and $f(x) - u$ is not decreasing in A_L^* and not increasing in A_R^*. We shall prove that $Q(x,u) = 0$ has a unique admissible solution such that $(x,u) \in B \in \{A_L^*, A_R^*\}$. Moreover the iteration scheme $M(x,u_{n+1}) = N(x,u_n)$ produces a fast convergence to the solution for $(x,u_n) \in B$.

B.1 Some Basic Properties

The functional Q is well defined on the set of admissible feedback controls. The problems stated in this paper are transferred to the problem of finding a feedback policy $u^*(x)$ such that $Q = 0$ and $(x, u^*(x)) \in A^*$. We will state and prove some basic inequalities and monotonicity properties satisfied by the functional on A_L^* and A_R^*.

Lemma 1 *Basic Inequalities*

(25) $$\{f(x) - v\}\,(\Pi_u - \Pi_v) \le M(x,u) - M(x,v) \le \{f(x) - u\}\,(\Pi_u - \Pi_v)\,.$$

hold for all (x,u) and (x,v) in A^* and Π_w is short for $\frac{\partial}{\partial w}\Pi(x,w)$. The function $M(x,u)$ is positive semidefinite.

Proof: These inequalities follow directly from the definition of M and the properties of Π. From Eq. (4) we obtain

$$M(x,u) - M(x,v) = \Pi(x,u) - \Pi(x,v) + \Pi_u(x,u)\,\{f(x) - u\} - \Pi_v(x,v)\,\{f(x) - v\}\ ,$$

and the concavity of Π implies

$$\Pi_u(x,u)(u - v) \le \Pi(x,u) - \Pi(x,v) \le \Pi_v(x,v)(u - v)$$

and Eq. (25) is obtained. This proves the first part of the lemma. Putting $v = f(x)$ and using $M(x, f(x)) \equiv 0$ yields the semidefinite property of M.$\Box$

We can now state and prove the key monotonicity properties of the operators involved.

Proposition 1 *Main Monotonicity Properties*
$M(x,u)$, $N(x,u)$ and $Q(x,u)$ are all monotone in u on $B \in \{A_L^*, A_R^*\}$
Moreover,

1. $(x - x_\delta^*)M(x,u)$ *is non-decreasing in* u *on* A^*,
2. $(x - x_\delta^*)N(x,u)$ *is non-increasing in* u *on* A^*,
3. $(x - x_\delta^*)Q(x,u)$ *is non-decreasing in* u *on* A^*.

Proof
We have $M_u(x,u) = \Pi_{uu}(x,u)(f(x) - u)$ and assumption A1 implies $M_u \leq 0$ if $u \leq f(x)$ and $M_u \geq 0$ if $u \geq f(x)$.Hence M is monotone in B implying that $(x - x_\delta^*)M(x,u)$ is nondecreasing in A^*.
Let Δ_{uv} be defined by

$$\Delta_{uv} = \Delta_{uv}(x) \overset{def}{=} \Pi_u(x, u(x)) - \Pi_v(x, v(x)) \,. \tag{26}$$

The concavity property of Π implies that $\Delta_{uv} \leq 0$ when $u \geq v$ and vice versa. The definition of the functional $N(x,u)$ in Eq. (10) implies

$$(x - x_\delta^*)[N(x,u) - N(x,v)] = \delta(x - x_\delta^*) \int_{x_\delta^*}^{x} \Delta_{uv}(s)ds \tag{27}$$

It follows directly that the left hand side of Eq. (27) has the sign of Δ_{uv} when $v \geq u$. Hence $(x - x_\delta^*)N(x,u)$ is non-increasing on A^*. The properties of Q follow from $Q = M - N$.□
We continue by investigating the iteration scheme and prove the general convergence result given by Theorem 1.

B.2 Iteration Scheme

We are studying the equation $Q(x,u) = 0$, or

$$M(x,u) = N(x,u) \quad \text{for} \quad (x,u) \in A^* \,. \tag{28}$$

Introducing M as given by Eq. (4) has several advantages. The function $M(x,u)$ measures a potential gain of changing the state or moving in state space, i.e., there can be a beneficial gain associated with changing the current state. It plays a role similar to the kinetic energy in a mechanical system. The motion by itself has the potential of doing a physical work which can be viewed as advantageous. The functional $N(x, u)$ as given by Eq. (10), can be associated with potential energy. It has the potential to change the current state of motion. The concavity property of $\Pi(x, u)$ with respect to u, implies uniqueness of the solution of Eq. (28) for $(x, u) \in A^*$.

Proposition 2 *Uniqueness*
Eq. (28) has at most one admissible solution.

Proof
Applying Eqs. (25-28) we get

$$\{f(x) - v(x)\}\Delta_{uv}(x) \le \delta \int_{x_\delta^*}^{x} \Delta_{uv}(s)ds \le \{f(x) - u(x)\}\Delta_{uv}(x). \tag{29}$$

Assume contrary to the proposition that u and v are two different solutions. If $u \neq v$ in A^* there exists an interval I bounded by $x_1 \neq x_\delta^*$ and x_δ^* such that $u \ge v$ in I and $u > v$ in a nonempty interval $J \subset I$. Let $x = z$ be the center point in J. It follows that $\Delta_{uv}(x) \le 0$ in I and strictly negative in J. We apply the natural restriction $\delta \ge 0$.
In A_L^* we have $f(z) \ge u(z) > v(z)$ implying

$$\{f(z) - v(z)\}\Delta_{uv}(z) < 0, \quad \delta \int_{x_\delta^*}^{z} \Delta_{uv}(s)ds \ge 0 \text{ and } \{f(z) - u(z)\}\Delta_{uv}(z) \le 0.$$

In A_R^* we have $u(z) > v(z) \ge f(z)$ and thereby

$$\{f(z) - v(z)\}\Delta_{uv}(z) \ge 0, \quad \delta \int_{x_\delta^*}^{z} \Delta_{uv}(s)ds \le 0 \text{ and } \{f(z) - u(z)\}\Delta_{uv}(z) > 0.$$

The proposition follows by noticing that both sets of inequalities above contradict the relations in (29).□
In order capture some inherent properties of the structure defined in Eq. (28) we introduce the concepts of super and sub solutions.

Definition 2 *Super and Sub Solutions*
An admissible control path in $B \in \{A_L^*, A_R^*\}$ *defined by* $u = u(x)$ *such that* $u(x_\delta^*) = f(x_\delta^*)$ *is called a super solution if* $(x - x_\delta^*)\,Q(x, u) \ge 0$ *and a sub solution if* $(x - x_\delta^*)\,Q(x, u) \le 0$.

The monotonicity of the defining functional (Proposition 1) makes it a generalized ordering of admissible feedback policies. It ensures that all admissible policy-paths are super solutions if they are above the first best policy and sub solutions if they are below, i.e. geometrical super (sub) solutions are formal super (sub) solutions.

A main input to a model is the statement about (natural) growth or change. It is the function $f(x)$ in the present context. This key function is a super-sub function meaning that it is super on the left side (A_L^*) and sub on the right side (A_R^*) as can be directly verified by noticing that according to assumption A2 we have $\psi(x) \geq 0$ and hence

$$(x - x_\delta^*)Q(x, f(x)) = -(x - x_\delta^*)\psi(x) \begin{cases} \geq 0, & x < x_\delta^* \\ \leq 0, & x > x_\delta^* \end{cases} .$$

Indeed, this makes perfect sense given that any admissible policy is bounded from above by $f(x)$ in A_L^* and from below by $f(x)$ in A_R^* and hence implying that $f(x)$ is a geometrical super-sub solution. A feedback solution which is both a super and sub solution on B must coincide with the unique solution u^*.
We will later (Lemma 5) provide a result that makes it unnecessary to distinguish between formal and geometric super or sub solutions.
In the next lemma we define main operators and summarizes their monotonicity.

Lemma 2 *Core Monotonicity Properties*
The following properties holds for admissible controls

1. $T_M(u) \overset{def}{=} (x - x_\delta^*) \cdot M(x, u)$ *is non-decreasing in* u.

2. $T_N(u) \overset{def}{=} (x - x_\delta^*) \cdot N(x, u)$ *is non-increasing in* u.

3. $T(u) \overset{def}{=} (x - x_\delta^*) \cdot Q(x, u)$ *is non-decreasing in* u.

This lemma is just a restatement of Proposition 1.□
We now define our solution procedure as an iteration scheme.

Definition 3 *Iteration Scheme and Short Notation*
Let $\{u_n\}$ *be a sequence of admissible policies. We denote*

(30) $$M_n \overset{def}{=} M(x, u_n) \quad \text{and} \quad N_n \overset{def}{=} N(x, u_n)\,.$$

Iterations or iteration scheme is short for any admissible sequence defined by

(31) $$M_{n+1} = N_n.$$

The iterations converge right away if $\delta = 0$. The exact solution is given by the first iterate as the solution of the ordinary algebraic equation $M(x, u_1) = S(x_0^*) - S(x)$. Basic structure in the iteration scheme is revealed in next lemmas and propositions.

Lemma 3 *Basic Alternating property*
Let $B \in \{A_L^*, A_R^*\}$ and $x \neq x_\delta^*$. Further let $\diamond$ represent a weak or strong inequality relation. Then $u \diamond v$ implies $V \diamond U$ where U and V are the first iterates of u and v. Moreover there is at most one iterate for each given admissible function.

Proof
The result is a direct consequence of the monotonic properties of T_M and T_N:

$$T_M(V) = T_N(v) \diamond T_N(u) = T_M(U) \quad \Rightarrow \quad V \diamond U.$$

We have already pointed out that $M(x, U)$ is strictly monotone in U for $U \neq f(x)$ in B. It means that the equation $M(x, U) = K(x) \geq 0$ has at most one solution $U = U(x)$. The case $K(t) = 0$ for $x = t$ is trivially given by $U(t) = f(t)$. The implicit function theorem ensures that $M(x, U) = N(x, u) > 0$ generates a function $U = U(x)$ which is piecewise smooth for any admissible function u. $\square$

Proposition 3 *Alternating Sub-Super Sequences*
The following properties hold for iterations in $B \in \{A_L^*, A_R^*\}$:

1. *If there exist two consecutive elements $u_{n+1} \geq u_n$ (less or equal), then u_{n+1} is a super (sub) solution and u_n is a sub (super) solution.*
2. *If an element in the sequence is a super or sub solution, the rest of the sequence will alternate between sub and super solutions.*

Proof
We will extensively utilize the following relations among the operators defined in Lemma 2:

$$(32) \quad T_M(v) = T_N(u) \quad \Rightarrow \quad T(u) = T_M(u) - T_M(v) \quad \text{and} \quad T(v) = T_N(u) - T_N(v)$$

The first part or premise states that v is the iterate of u. Both implied relations follow directly from the definitions of the iteration and the operators. Notice that the first implied relation is a function of two points in A^* while the second relation is not. The latter depends on two paths. Inserting $v = u_{n+1} \geq u_n = u$ in the first relation implies

$T(u) \leq 0$ and the second relation implies $T(v) \geq 0$ or signs reversed if $u_{n+1} \leq u_n$. This proves statement 1.
Let $u = u_n$ be a super (sub) solution. The first implied relation in Eq. (32) gives $T_M(u) \geq T_M(v)$ (or reversed inequality). This relation holds pointwise and the monotonicity of T_M ensures that $v = u_{n+1}$ is a sub (super) solution. Statement 2 is established by reapplying the argument.$\square$
Next lemma reveals a potential structure in super/sub sequences. They may contain two subsequences of opposite monotonicity. The pattern is visualized in figure 3. More precisely it reads:

Lemma 4 *Increasing and Decreasing Subsequences*
Let $u_0 > u_2$ for $x \neq x^*_\delta$. Then $u_{2n} > u_{2n+2}$ and $u_{2n+1} < u_{2n+3}$ for $x \neq x^*_\delta$. In addition, if an element is a super or sub solution the two subsequences $\{u_{2n}\}$ and $\{u_{2n+1}\}$ are bounded by each other.
The results hold if all inequalities are reversed or in weak form.

Proof
Applying Lemma 3 to u_n and u_{2+n} for $n \in \{0, 1, 2, 3, \ldots\}$ for each inequality relation establishes the two subsequences with their monotonicity properties. We demonstrate it for the relational operator ">":
Direct use of Lemma 3 yields $u_0 > u_2 \Rightarrow u_1 < u_3 \Rightarrow u_2 > u_4 \Rightarrow u_3 < u_5 \ldots$ implying $u_0 > u_2 > u_4 > \ldots$ and $u_1 < u_3 < u_5 \ldots$
Without loss of generality we may assume that u_0 is a super solution and the sequence is alternating between super and sub solutions according to Proposition 3. Hence $u_0 > u_2 > u_4 > \ldots$ is a sequence of super solutions and $u_1 < u_3 < u_4 < \ldots$ a sequence of sub solutions. Moreover, $u_{2n} \geq u_{2n+1}$ and the boundedness is established. Thus both sequences are monotone and bounded and hence they converge.$\square$
The next lemma allows us to focus on geometric super or sub solution. All formal super (sub) solutions become geometric sub (super) solutions after a single iteration. This result is indeed important. It implies that the first best solution is always between consecutive iterates of a formal super (sub) seed that we know from Proposition 3 produces an alternating sequence of sub and super solutions. There is now available a strait forward way to test for super/sub properties and produce feedback policies on either side of the unknown first-best solution.

Lemma 5 *Formal and Geometrical Super or Sub Solutions*

A formal super or sub solution becomes a geometric sub or super solution after a single iteration in $B \in \{A_L^*, A_R^*\}$.

Proof
Let $v = v(x)$ be the iterate of $u = u(x)$ and $u^* = u^*(x)$ be the optimal feedback solution. Applying Eq. (26) and assuming $x \neq x_\delta^*$ we define $\eta(x)$ by

$$\eta(x) \stackrel{def}{=} T_M(v) - T_M(u^*) = T_N(u) - T_N(u^*) = \delta(x - x_\delta^*) \int_{x_\delta^*}^{x} \Delta_{uu^*}(s)ds$$

Let us investigate a crossing between v and u^* at $x = z$. Crossing means that there is a $x = z$ such that $v \neq u^*$ immediately to the right in A_R^* and to the left in A_L^* and hence $\eta(x) \neq 0$ for x in an nonempty interval J. In this interval we have $\Delta_{uu^*}(x) \neq 0$ and the sign of η in J is given by

$$\begin{array}{rcl} \dfrac{\eta(x)}{\delta(x - x_\delta^*)} & = & \dfrac{\eta(z)}{\delta(x - x_\delta^*)} + \displaystyle\int_z^x \Delta_{uu^*}(s)ds = \int_z^x \Delta_{uu^*}(s)ds \\ & \Downarrow & \\ \operatorname{sgn}(\eta) & = & \operatorname{sgn}(\Delta_{uu^*}). \end{array} \tag{33}$$

Let u be a super solution and thereby v is a sub solution in accordance with Proposition 3. This implies that $v \leq u$. If v is not a geometric sub solution, it must cross u^* from below moving away from x_δ^*. Otherwise v will be a super solution in the neighborhood of x_δ^*! Hence we have an interval J where $u \geq v > u^*$ and thereby $\Delta_{uu^*} < 0$ in the interval. The definition of η and Eq. (33) imply $T_M(v) < T_M(u^*)$ in J which is contradicting the assumption that J exists! The proof in the case with u being a sub solution is similar. □

B.3 Convergence of Iterations

The iterations converge under quite general conditions. In this section we use a constructive approach to prove this convergence. We start by proving the convergence for a particular sequence starting from a geometric super-sub solution. Since there is no convergence problems if $\delta = 0$, we consider only the case $\delta > 0$.
Let the function ρ be defined by the expression

$$\rho(x) = \frac{\delta |\int_{x_\delta^*}^{x} m_2(s)ds|}{m_1(x)|f(x) - f_2(x)|} \tag{34}$$

where f_2 is the second iterate of $f(x)$ and m_1 and m_2 are defined in assumption A1. The

expression need not be defined for all $x \in X$.
The seed $u_0 = f(x)$ generates a special super-sub sequence $\{f_0 = f,\ f_1,\ f_2,\ \ldots\}$ that can be shown to converge under mild conditions. We state the result in the next proposition.

Proposition 4 *Convergence of a Special Sequence*
The special sequence $\{f_0, f_1, f_2 \ldots\}$ produced by $f_0 = f(x)$ in the iteration scheme converges to the unique solution u^* if $\rho(x) < 1$. The elements in the sequence are consecutive geometric super and sub solutions and the unique solution is approximated by

$$u^* = \frac{1}{2}(f_n + f_{n+1}) + \epsilon \quad \text{where} \quad |\epsilon| \leq \frac{1}{2}|f_n - f_{n+1}| \,.$$

Proof
The construction of the admissible sequences ensures convergence at $x = x^*_\delta$ and we are therefor only considering $x \neq x^*_\delta$ in the proof. We have in A^*_L that $f_0 > f_2$ and in A^*_R that $f_0 < f_2$. The weak inequalities follow directly from the definition of A^*. The strict inequalities come from

$$T_M(f_2) = T_N(f_1) = (x - x^*_\delta) \cdot \psi(x) \neq 0 = T_M(f) \quad \text{hence} \quad f_2 \neq f.$$

The nonzero condition is a result of assumption A2. By Lemma 4 we have two monotone sequences $f_0 > f_2 > f_4 > \ldots$ and $f_1 < f_3 < f_5 < \ldots$ in A^*_L, where the first is a sequence of geometric super solution and the second a sequence of geometric sub solutions. Both sequences are monotone and bounded and hence they converge. The same holds in A^*_R with reversed inequalities. We need to show that they converge to the same limit[13], that is, ruling out the existence of a period two cycle in the iteration scheme. It is done here by proving that the iterations represent a contraction scheme. We have

$$\begin{aligned}
|M_{n+1} - M_n| = |N_n - N_{n-1}| &= \delta \left| \int_{x^*_\delta}^{x} [\Pi_u(s, f_n(s)) - \Pi_u(s, f_{n-1}(s))] \right| \\
\Leftrightarrow |M_u(x, \theta_n)(f_{n-1} - f_n)| &= \delta \left| \int_{x^*_\delta}^{x} \Pi_{uu}(s, \phi_{n-1}(s))(f_n(s) - f_{n-1}(s))\, ds \right| \\
\Leftrightarrow |(f - \theta_n)\Pi uu(x, \theta_n)(f_{n-1} - f_n)| &= \delta \left| \int_{x^*_\delta}^{x} \Pi_{uu}(s, \phi_{n-1}(s))(f_n(s) - f_{n-1}(s)) ds \right| .
\end{aligned}$$

The mean value theorem has been applied. θ_n and ϕ_n are between f_n and f_{n+1}. From the first part of the proof we know that $|f - \theta_n| \geq |f - f_2|$ for $n \geq 1$. Applying assumption A1

[13]The approximation formula holds even if there are no convergence to the same limit because u^* is always between consecutive iterates.

we obtain

$$|f - f_2| \, m_1 \, |f_{n+1} - f_n| \leq \delta \left| \int_{x_\delta^*}^{x} m_2 \cdot (f_n - f_{n-1}) \, ds \right| \leq \delta \left| \int_{x_\delta^*}^{x} m_2 \, ds \right| \cdot \|f_n - f_{n-1}\| \, .$$

The maximum norm have been applied. In particular we obtain at a $x \neq x_\delta^*$ where $|f_{n+1} - f_n| = \|f_{n+1} - f_n\|$ the following

$$|f - f_2| \, m_1 \, \|f_{n+1} - f_n\| \leq \delta \left| \int_{x_\delta^*}^{x} m_2 \, ds \right| \cdot \|f_n - f_{n-1}\| \;\; \text{or} \;\; \|f_{n+1} - f_n\| \leq \rho \|f_n - f_{n-1}\| \, .$$

This shows that the particular sequence is contracting.□
In figure 3 we illustrate a typical way that our policy sequences converge. Finally the result

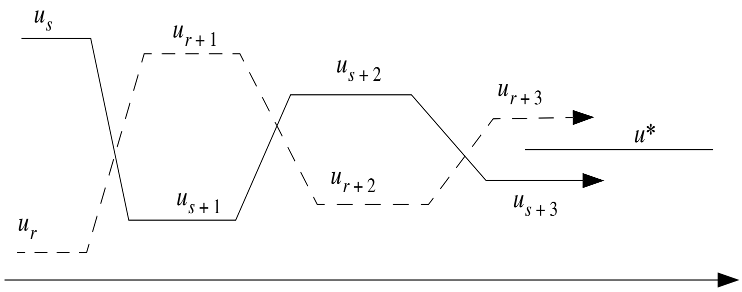

Figure 3: This figure shows schematically how the iterations work when started from a formal super or sub seed with new iterations along the horizontal axis. Notice that the sequence $\{u_s, u_{s+2}, u_{s+4}, \cdots\}$ is monotonically decreasing whereas $\{u_r, u_{r+2}, u_{r+4}, \cdots\}$ is monotonically increasing, converging to the same limit u^*.

can be generalize to an arbitrary sequence in A^*.

Proposition 5 *General Convergence Result*
Assume that the special sequence starting with $u_0 = f(x)$ converges. Then all admissible infinite sequences converge.

Proof
Assume first that there exist an iterate u_k bounded by $f = f_0$ and f_1. This implies by repeated use of the weak inequality form of Lemma 3 that u_{k+n} is bounded by f_n and f_{n+1} and hence converge. Assume that there is an iterate u_k not bounded by f_1. Let us consider an admissible policy $\tilde{u}_1$ that bounds both u_k and f_1. On the left of the reference point we must have $\tilde{u}_1 \leq f_1 \Rightarrow T_N(\tilde{u}_1) \geq T_N(f_1) \Rightarrow T_M(\tilde{u}_2) \geq T_M(f_2) \Rightarrow \tilde{u}_2 \geq f_2$ and reversed inequalities to the right of the reference point. This means that the iterates of $\tilde{u}_2$ are

squeezed by the iterates of f_0 and f_2 and hence the sequence $\{\tilde{u}_1, \tilde{u}_2, \tilde{u}_3, \ldots\}$ converge. In turn this implies that the sequence with an element not bounded by f_0 and f_1 must converge since it is squeezed between the iterates of f_0 and $\tilde{u}_1$.$\square$

The last proposition ensures the general statement about convergence given in theorem 1. The error bound given in the theorem must hold by the simple fact that the sequence $\{v_n\}$ is an alternating geometrical super-sub sequence which implies that the optimal feedback solution is between two consecutive iterations. This completes the proof of our main result.